Integration by Trigonometric Substitu

The following trigonometric substitution are adapted in the integra

Form of integrand	Substitution
1) $\sqrt{a^2-x^2}$,	$x = a\sin\theta$ or $x = a\cos$
2) $\sqrt{x^2+a^2}$,	$x = a\tan\theta$ or $x = a\cot$
3) $\sqrt{x^2-a^2}$,	$x = a\sec\theta$ or $x = a\,\text{cos}$

Some standard Results

(1) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

(2) $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

(3) $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$$

(4) $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$$

(5) $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

(6) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$

(7) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln\left|x + \sqrt{x^2 + a^2}\right| + c$

(8) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln\left|x + \sqrt{x^2 - a^2}\right| + c$

(9) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x - a}{x + a}\right| + c$

(10) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right| + c$

1. Q. Prove that $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

A- L.H.S $\int \frac{dx}{\sqrt{a^2 - x^2}}$

Let $x = a\sin\theta$ $\Rightarrow \sin\theta = x/a$

$dx = a\cos\theta\, d\theta$ $\Rightarrow \theta = \sin^{-1} x/a$

$\int \frac{a\cos\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\, d\theta$

$= \int \frac{a\cos\theta}{\sqrt{a^2(1 - \sin^2\theta)}}\, d\theta$

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$$= \int \frac{a\cos\theta}{\sqrt{a^2\cos^2\theta}}\, d\theta$$

$$= \int \frac{a\cos\theta}{a\cancel{\cos\theta}}\, d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sin^{-1} x/a + c$$

$= R \cdot H \cdot S$ (Proved)

2Q. Evaluate $\int \frac{dx}{25 - x^2}$

A- $\int \frac{dx}{(5)^2 - (x)^2}$

Apply the formula $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right|$

So $\int \frac{dx}{(5)^2 - x^2}$

$$= \frac{1}{2 \times 5} \ln\left|\frac{5+x}{5-x}\right| + c$$

$$= \frac{1}{10} \ln\left|\frac{5+x}{5-x}\right| + c$$

3Q Evaluate $\int \frac{dx}{x^2+9}$

A- $\int \frac{dx}{(x)^2+(3)^2}$

Apply the formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$\therefore \int \frac{dx}{(x)^2+(3)^2}$

$= \frac{1}{3}\tan^{-1}\frac{x}{3} + c$

4Q Evaluate $\int \frac{dx}{\sqrt{x^2+16}}$

A- $\int \frac{dx}{\sqrt{(x)^2+(4)^2}}$

Apply the formula $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c$

$\therefore \int \frac{dx}{\sqrt{x^2+(4)^2}}$

$= \ln\left|x+\sqrt{x^2+(4)^2}\right| + c$

$= \ln\left|x+\sqrt{x^2+16}\right| + c$

5.Q Evaluate $\int \sqrt{16 - x^2}\, dx$

A- Apply the formula

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1} x/a + C$$

$$\therefore \int \sqrt{(4)^2 - (x)^2}\, dx$$

$$= \frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\left(x/4\right) + C$$

$$= \frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\left(x/4\right) + C$$

$$= x/2\sqrt{16 - x^2} + 8\sin^{-1}\left(x/4\right) + C$$

6.Q Evaluate $\int \frac{dx}{x\sqrt{25 - (\ln x)^2}}$

A- $\int \frac{dx}{x\sqrt{(5)^2 - (\ln x)^2}}$

Let $\ln x = t$

$$\Rightarrow \frac{1}{x}dx = dt$$

$$\therefore \int \frac{dt}{\sqrt{(5)^2 - (t)^2}}$$

$$= \sin^{-1}\frac{t}{5} + C$$

$$= \sin^{-1}\left(\frac{\ln x}{5}\right) + C$$

7.Q Evaluate $\int \frac{e^{3x}}{\sqrt{4-e^{6x}}}dx$

A- $\int \frac{e^{3x}}{\sqrt{(2)^2-(e^{3x})^2}}dx$

Let $e^{3x} = t$

$\Rightarrow 3 \cdot e^{3x} dx = dt$

$\Rightarrow e^{3x} dx = \frac{1}{3} dt$

$\therefore \frac{1}{3}\int \frac{dt}{\sqrt{(2)^2-(t)^2}}$

$= \frac{1}{3} \sin^{-1} \frac{t}{2} + C$

$= \frac{1}{3} \sin^{-1}\left(\frac{e^{3x}}{2}\right) + C$

8. Q Evaluate $\int \frac{e^{x} \cos e^{x}}{\sqrt{\sin^2 e^x + 9}}dx$

A- $\int \frac{e^x \cos e^x}{\sqrt{(\sin e^x)^2+(3)^2}}dx$

Let $\sin e^x = t$

$\Rightarrow e^x \cdot \cos e^x \, dx = dt$

$\therefore$ $\int \frac{dt}{\sqrt{(t)^2+(3)^2}} = \log\left|t+\sqrt{t^2+3^2}\right| + C$

$= \log\left|\sin e^x + \sqrt{\sin^2 e^x + 9}\right| + C$

9.Q Evaluate $\int \frac{dx}{\sqrt{11-4x^2}}$

Ans- N.B Always make the co-efficient of x^2 is 1.

$$\int \frac{dx}{\sqrt{4\left(\frac{11}{4}-x^2\right)}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{\sqrt{11}}{2}\right)^2-(x)^2}}$$

$$= \frac{1}{2} \times \sin^{-1}\frac{x}{\sqrt{11}/2} + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{11}}\right) + C$$

10.Q Evaluate $\int \frac{dx}{3x^2+7}$

Ans- $\int \frac{dx}{3\left(x^2+7/3\right)}$

$$= \frac{1}{3}\int \frac{dx}{(x)^2+\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{3} \times \frac{1}{\left(\sqrt{7}/\sqrt{3}\right)} \times \tan^{-1}\frac{x}{\sqrt{7}/\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}\sqrt{7}} \times \tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right) + C$$

Q11 Evaluate $\int \frac{dx}{x^2+6x+13}$

Ans $\int \frac{dx}{x^2+6x+(3)^2-(3)^2+13}$

[N.B. Make it perfect square.]

$= \int \frac{dx}{\{(x)^2+2\cdot 3\cdot x+(3)^2\}-9+13}$

$= \int \frac{dx}{(x+3)^2+4}$

$= \int \frac{dx}{(x+3)^2+(2)^2}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+c$

Q12 Evaluate $\int \frac{dx}{\sqrt{8-2x-x^2}}$

Ans $\int \frac{dx}{\sqrt{8-(x^2+2x)}}$

$= \int \frac{dx}{\sqrt{8-\{x^2+2x+(1)^2-(1)^2\}}}$

$= \int \frac{dx}{\sqrt{-(x^2+2x+1^2)+1+8}}$

$= \int \frac{dx}{\sqrt{-(x+1)^2+9}}$

$= \int \frac{dx}{\sqrt{(3)^2-(x+1)^2}} = \sin^{-1}\left(\frac{x+1}{3}\right)+c$

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Assignments

Q. Evaluate

(1) $\int \frac{e^{4x}}{e^{8x}+4}\,dx$

(2) $\int \frac{\sec x\cdot\tan x}{\sec^2 x+4}\,dx$

(3) $\int \frac{dx}{\sqrt{3x^2+4}}$

(4) $\int \frac{dx}{\sqrt{x^2-9}}$

(5) $\int \sqrt{25-x^2}\,dx$

(6) $\int \frac{dx}{\sqrt{x^2-4x+2}}$

(7) $\int \frac{\cos\theta}{\sqrt{4-\sin^2\theta}}\,d\theta$