*. INTEGRATION *

Integration By Parts

This method is most useful when $f(x)$ is a power of x and $g(x)$ can be integrated successively upto the stage after which the successive derivative of $f(x)$ becomes zero.

Let u and v be two functions of x, then for the integration of the product uv of the functions, we use the formula

$$\int uv\,dx = u\cdot\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$$

i.e The integral of the product of two functions = first f? x integral of the second f? − Integral of [Differential of first f? x integral of second f?]

The choice of first f?(u) can be done according to the order of the letters of the word "ILATE" where

I = Inverse Trigonometric function

L = Logarithmic function

A = Algebraic function

T = Trigonometric function

E = Exponential function

Example - 1

(a) ~~Integrate~~ $\int$

Integrand	first $f^n(u)$	Second $f^n(v)$
$x \cos x$	x	$\cos x$
$x^2 e^x$	x^2	e^x
$x^7 \ln x$	$\ln x$	x^7
$\ln x$	$\ln x$	1
$\tan^{-1} x$	$\tan^{-1} x$	1
$x \tan^{-1} x$	$\tan^{-1} x$	x

Q Integrate $\int x e^x dx$

Sol? $\int x e^x dx$

Here $u = x$ & $v = e^x$

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$$

$$= x\int e^x dx - \int\left[\frac{d}{dx}x\cdot\int e^x dx\right]dx$$

$$= xe^x - \int 1\cdot e^x dx$$

$$= xe^x - e^x + c$$

$$= e^x(x-1) + c$$

Q Integrate $\int x\cos x\,dx$

Sol? $\int x\cos x\,dx$

Here $u = x$ & $v = \cos x$

$$= x\int\cos x\,dx - \int\left[\frac{d}{dx}x\cdot\int\cos x\,dx\right]dx$$

$$= x\sin x - \int 1\cdot\sin x\,dx$$

$$= x\sin x - (-\cos x) + c$$

$$= x\sin x + \cos x + c$$

Q. Integrate $\int x^7 \ln x \, dx$.

Sol: $\int x^7 \ln x$

Here $u = \ln x$ & $v = x^7$

$$= \ln x \int x^7 dx - \int \left[\frac{d}{dx} \ln x \int x^7 dx\right] dx$$

$$= \ln x \cdot \frac{x^8}{8} - \int \frac{1}{x} \cdot \frac{x^8}{8} dx$$

$$= \frac{x^8 \cdot \ln x}{8} - \frac{1}{8} \int x^7 dx$$

$$= \frac{x^8}{8} \ln x - \frac{1}{8} \times \frac{x^8}{8} + C$$

$$= \frac{x^8}{8} \left(\ln x - \frac{1}{8}\right) + C$$

Q. Integrate $\int \tan^{-1} x \, dx$

Sol: Here $u = \tan^{-1} x$ & $v = 1$

$$\int \tan^{-1} x \, dx = \tan^{-1} x \int 1 dx - \int \left[\frac{d}{dx} \tan^{-1} x \int 1 dx\right] dx$$

$$= x \cdot \tan^{-1} x - \int \frac{1}{1+x^2} \cdot x \, dx$$

$$= x \cdot \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

2nd

2. Evaluate $\int \log(1+x^2)\,dx$

Here $u = \log(1+x^2)$, $v = 1$

$$\int \log(1+x^2)\cdot 1\,dx$$

$$= \log(1+x^2)\cdot\int 1\,dx - \int\left[\frac{d}{dx}\log(1+x^2)\cdot\int 1\,dx\right]dx$$

$$= \log(1+x^2)\cdot x - \int \frac{2x}{1+x^2}\cdot x\,dx$$

$$= x\log(1+x^2) - 2\int \frac{x^2}{1+x^2}\,dx$$ (divide) ⟸ Note

$$= x\log(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= x\log(1+x^2) - 2\int 1\,dx + 2\int \frac{1}{1+x^2}\,dx$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + C$$

Formula

✓

① $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C$

② $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

③ $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

Q Integrate $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

Soln By using the formula

$$\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + C$$

Here $f(x)=\frac{1}{x}$ & $f'(x)=-\frac{1}{x^2}$

So $\int e^x\left[\frac{1}{x}+-\left(\frac{1}{x^2}\right)\right]dx$

$$= e^x \times \frac{1}{x} + C$$

Q Integrate $\int e^{3x}\cos 2x\,dx$

Soln By using the formula.

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c$$

Here $a=3$, $b=2$.

So $\int e^{3x}\cos 2x\,dx = \frac{e^{3x}}{3^2+2^2}[3\cos 2x+2\sin 2x]+c$

This formula is derived by using by parts formula of Integration.

Q Integrate $\int x^2 e^x dx$

Sol^n $$\int x^2 e^x dx = x^2 \int e^x dx - \int \left[\frac{d}{dx} x^2 \int e^x dx\right] dx$$

$$= x^2 e^x - \int 2x \cdot e^x dx$$

$$= x^2 e^x - 2\int x e^x dx$$

$$= x^2 e^x - 2\left[x \int e^x dx - \int \left\{\frac{d}{dx} x \int e^x dx\right\} dx\right]$$

$$= x^2 e^x - 2\left[x e^x - \int 1 \cdot e^x dx\right]$$

$$= x^2 e^x - 2\left[x e^x - e^x\right] + c$$

$$= x^2 e^x - 2x e^x + 2e^x + c$$

Assignments

Q Integrate

(1) $\int x \operatorname{cosec}^2 x \, dx$

(2) $\int x \sin x \, dx$

(3) $\int \sin^{-1} x \, dx$

(4) $\int e^x \sin x \, dx$

(5) $\int \ln x \, dx$

(6) $\int x \tan^{-1} x \, dx$

(7) $\int e^{2x} \sin 3x \, dx$

(8) $\int \frac{e^x}{x}(1 + x \ln x) dx$

(9) $\int x^2 \cos x \, dx$

(10) $\int x^5 \ln x \, dx$